The friction angle and critical state void ratio of sands

L´angle de frottement et l'indice des vides des sables

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ABSTRACT

The friction angle of sands is computed by Bolton as the sum of the critical state friction angle and a dilatancy term which is a function of mean effective pressure and void ratio. Critical state is reached when dilatancy vanishes, either due to volume change – in drained shear – or effective pressure change – in undrained shear. Therefore, equating Bolton's dilatancy term to zero yields, at least theoretically, an implicit relationship between mean pressure and the critical state void ratio of sands. It is found that this relationship yields unrealistic results, mainly because Bolton's expression if of phenomenological nature and was not intended to be used for this purpose. In this paper, a minor modification to Bolton's dilatancy term is proposed. It is proved that the modified expression has the capacity to predict both the peak friction angle and the critical state void ratio for any void ratio and effective pressure within the range of engineering interest.

RÉSUMÉ

L´angle de frottement des sables est calculé par Bolton comme la somme de l´angle de frottement dans l´étàt critique et un terme de dilatance, qui est une fonction de la pression effective moyenne, et l´indice des vides. L´étàt critique est atteint quand la dilatance disparait à cause, soit du changement de volume – dans le cisaillement drainée – soit d´un changement de pression effective – dans le cisaillement non drainée. Donc, en égalant à zéro le terme de dilatance de Bolton on obtient, au moins théoriquement, une relation implicite entre la pression moyenne et l´indice des vides dans l´étàt critique des sables. On voit que cette relation fournit des résultats irréalistes, surtout parce que l´expression de Bolton est de nature phénoménologique et elle n´a pas été pensée pour être employée à cette fin. Dans cet article, une modification mineur du terme de dilatance de Bolton est proposée. On prouve que l´expression modifiée a la capacité de prédire soit l´angle de frottement interne et l´indice des vides dans l´étàt critique des sables pour chaque indice des vides et pression effective dans le rang d´intérêt de l´ingénierie.

Keywords : sands – critical state void ratio – peak friction angle – dilatancy

1 INTRODUCTION

Research on the shear strength of sands for practical applications has two main branches: the first one focuses on the prediction of the peak friction angle of dilating sands, while the second one focuses on the prediction of undrained shear strength of loose sands.

Pressure and void ratio dependence of shear strength is acknowledged for in both research fields. For drained shear, some outstanding contributions are (Bolton 1986, de Beer 1965, Lee & Seed 1967, Maeda & Miura 1999a, Maeda & Miura 1999b, Marsal 1967). For undrained shear, main contributions are (Been & Jefferies 1985, Been et al 1991, Castro 1975, Castro & Poulos 1977, Ishihara 1993, Poulos 1981, Verdugo & Ishihara 1996).

For drained shear of dilating sands, it is a common practice to compute the peak friction angle ϕ as the sum of the critical state friction angle ϕ_c and a dilatancy term ψ which in turn depends on void ratio e and effective mean pressure p, or

$$\phi = \phi_c + \psi\left(p,e\right) \tag{1}$$

The most widely used expression in the form of eqn. (1) is that of Bolton (1986) which can be put in the form

$$\phi = \phi_c + \Delta\phi D_r\left(Q - \ln\left(p/p_{ref}\right)\right) - R \tag{2}$$

where $\Delta\phi$=3°, R=3°, Q is a fit parameter and p_{ref} is a reference pressure, taken equal to 1 kPa by Bolton (1986).

Critical state is reached when dilatancy vanishes; a critical state void ratio e_c is defined at the critical state (Casagrande 1936, Casagrande 1975, Ishihara 1993, Núñez 1991). It is found that e_c depends on mean pressure (Casagrande 1975, Castro 1975, Castro & Poulos 1977 , Poulos 1981 , Ishihara 1993, Verdugo & Ishihara 1996), a fact that can be put in a quantitative form by the implicit relationship

$$\psi\left(p,e_c\right) = 0 \tag{3}$$

On the other hand, research on the undrained behavior of loose sands has shown that the state parameter (Been & Jefferies 1985)

$$\Psi = e - e_c \tag{4}$$

can be used to predict the undrained shear strength of a given sand (Been 6 Jefferies 1985, Been et al 1991). Moreover, constitutive models for sands has been developed around the state parameter (e.g. Jefferies 1993) and even strain softening behavior has been connected to eqn. (4) (Muir Wood et al 1994). Other similar state parameters have also been defined (e.g. Cubrinovski & Ishihara 1998).

Theoretically, eqn. (3) can be used to compute e_c, and therefore the set of eqns. (1) to (4) should suffice to predict the shear strength of sands in the full range of pressure and void ratio of engineering interest, for both drained and undrained conditions.

However, this is not the case, mainly because eqn. (2) is of phenomenological nature and was not intended to be used to compute e_c. In the following sections, a minor modification to eqn. (2) is derived. With this modification, Bolton's expression can be used within the general framework of critical state soil models, improving their usability due to a better estimation of the peak friction angle and critical state void ratio of sands for all pressures and void ratios encountered in practice.

2 MEANING OF PARAMETER *Q* IN BOLTON'S EQUATION

In eqn. (2), Q is a material parameter that depends on the crushing resistance of sand particles (Bolton 1986). To highlight the significance of Q, eqns. (1) and (2) are combined to yield (Sfriso 2007, Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008)

$$\psi = -\Delta\phi D_r \ln(\chi_B) - R \tag{5}$$

where

$$\chi_B = \frac{p}{\exp(Q)\ p_{ref}} \tag{6}$$

acts as a stress level index. Theoretically, χ_B is limited by the condition ψ=0° or

$$\chi_B = \exp\left(\frac{-R}{\Delta\phi D_r}\right) \tag{7}$$

Equating eqn. (6) and (7), the relationship between the critical void ratio and mean pressure can be computed to be

$$e_c = e_{\max} - \frac{(e_{\max} - e_{\min})R}{\Delta\phi\left(Q - \ln\left(p/p_{ref}\right)\right)} \tag{8}$$

where e_{max} and e_{min} are the max. and min. void ratios, respectively. The rest of the parameters being fixed, it is therefore concluded that parameter Q controlls the shape of the $e_c - p$ relationship.

3 MODIFIED DILATANCY TERM

The stress level index introduced by eqn. (6) is void ratio independent. However, particle crushing is known to be void ratio dependent (e.g. Pestana & Whittle 1995). For isotropic compression test paths, Pestana (Pestana & Whittle 1995, Pestana et al 2002) proposed the expression

$$p_{ult} = e_0^{-\frac{1}{\rho_c}} p_r\, p_{ref} \tag{9}$$

where p_r and ρ_c are material parameters. Eqn. (9) can be used to define a new stress level index (Sfriso 2007, Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008)

$$\chi = p/p_{ult} \tag{10}$$

which may replace χ_B in eqn. (5) to yield a new dilatancy term of the form

$$\psi = -\Delta\phi D_r \ln(\chi) - R \tag{11}$$

Fig. 1 shows isotropic compression test results of Toyoura Sand (Pestana & Whittle 1995) and some iso-χ and iso-χ_B lines. Following Pestana (Pestana & Whittle 1995), p_r=55 was adopted for Toyoura Sand. While Pestana reports that ρ_c varies between 0.33 and 0.45, a constant value ρ_c=0.40 is adopted for convenience in the rest of this paper.

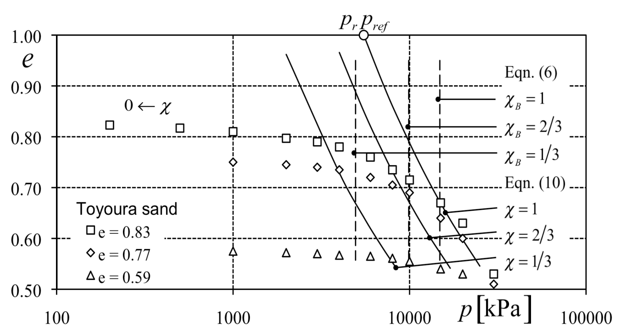

Figure 1. Isotropic compression of Toyoura sand (Pestana & Whittle 1995) and lines of constant χ and χ_B.

Calibration of eqn. (11) with the data used by Bolton to support eqn. (2) yields $\Delta\phi$=3° and R=2°, used here as default parameters. Fig. 2 shows ψ for Sacramento River Sand (data from Lee & Seed 1967) and Toyoura Sand (data from Bolton 1987, Fukushima & Tatsuoka 1984) and the predictions by eqn. (11). It may be noted that the predictive capability of eqn. (2) is retained by eqn. (11).

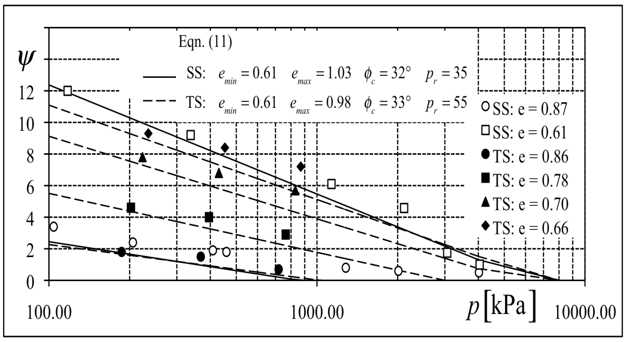

Figure 2. Dilatancy term ψ for Toyoura sand (TS) (data from Bolton 1987, Fukushima & Tatsuoka 1984), Sacramento River sand (SS) (data from Lee & Seed 1967), and predictions by eqn. (11).

4 MODIFIED EXPRESSION FOR THE PEAK FRICTION ANGLE

In turn, eqn. (11) can be inserted in eqn. (1) to yield the final expression

$$\phi = \phi_c - \Delta\phi D_r \ln(\chi) - R \tag{12}$$

Bolton (1986, 1987) limited the validity of eqn. (2) to p>150kPa to avoid overestimation of dilatancy. This limitation also applies to eqn. (12).

5 CRITICAL STATE VOID RATIO

To compute the critical state void ratio, eqn. (11) is combined with eqns. (9) and (10) to yield the following implicit $e_c - p$ relationship (Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008)

$$p\big|_{e_c} = \frac{p_r}{e_c^{2.5}} \exp\left(\frac{-R}{\Delta\phi\ D_{r,c}}\right) p_{ref} \tag{13}$$

where

$$D_{r,c} = \frac{e_{max} - e_c}{e_{max} - e_{min}} \tag{14}$$

Fig. 3 shows the $e_c - p$ line of Toyoura sand (data from Verdugo & Ishihara 1996), the prediction by eqn. (8) and the prediction by eqn. (13). It may be observed that eqn. (13) shows a much better agreement with experimental data than eqn. (8), using with the same number of material parameters (Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008). A more accurate fit, if required, may be achieved by fine tuning $\Delta\phi$ and R for a particular data set, as indicated in Fig. 13.

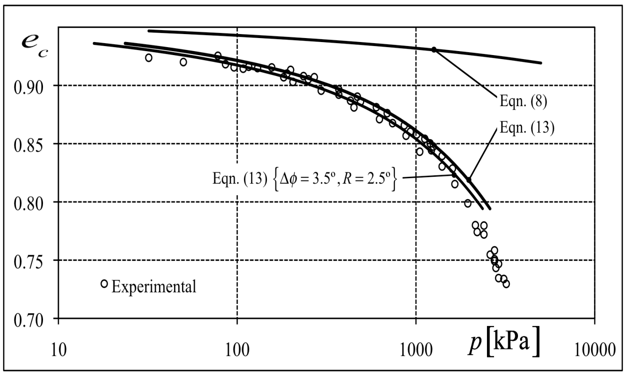

Figure 3. $e_c - p$ relationship for Toyoura sand (data from Verdugo & Ishihara 1996) and the prediction by eqns. (8) and (13) (Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008).

The critical state void ratio e_c computed by eqn. (13) can be used to compute the state parameter Ψ defined by eqn. (4) with enough accuracy for engineering analyses. Therefore, a single set of parameters – namely e_{min}, e_{max}, p_r and ϕ_c - defines the functional relationships $\phi(p,e)$ and $e_c(p)$, thus providing an accurate prediction of peak friction angle for dilating sands while keeping the predictive capability of the critical state models like Nor-Sand (Jefferies 1993) for undrained shear of loose sands. A complete constitutive model equipped with eqn. (11) expressions can be found in (Sfriso 2008a, Sfriso 2008b, Sfriso & Weber 2008).

6 CONCLUSIONS

A modification to Bolton's expression for the peak friction angle of sands is proposed. This modified expression, when equated to zero, yields a relationship between the critical state void ratio and mean pressure which is accurate enough for routine engineering analysis and that can be used to predict the undrained strength of loose sands. The proposed approach is convenient for constitutive modelling of sands because it allows the prediction of the shear strength of a given sand as a function of void ratio and mean pressure using a fixed set of material parameters which is independent of stress state, void ratio, drainage conditions or stress path.

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